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Invariants for the time-dependent harmonic oscillator: I

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Abstract. Two equivalent families of linear invariants and two equivalent families of quadratic invariants are obtained for a harmonic oscillator with variable mass or with variable frequency. Explicit formulae are obtained for the linear and quadratic invariants in the case of a damped oscillator and the Hamiltonian is seen as a special quadratic invariant.

1. Introduction

A second-degree invariant for the variable-frequency oscillator with Hamiltonian

$$H(t)=\tfrac{1}{2}p^2/M_0+\tfrac{1}{2}M_0\omega^2(t)q^2, \tag{1.1}$$

associated with the problem of a slowly lengthening pendulum, is well known (Lewis 1967, 1968, Lewis and Riesenfeld 1969, Wollenberg 1980). The invariant has the form

$$I(t)=q^2/\sigma^2+(\dot{\sigma}q-\sigma p/M_0)^2, \tag{1.2}$$

where $\sigma(t)$ is any solution of the Pinney equation (Pinney 1950)

$$\ddot{\sigma}+\omega^2(t)\sigma=1/\sigma^3. \tag{1.3}$$

Lutzky (1978) has related the Lewis invariant (1.2) to a Noether symmetry (Noether 1918) and an interesting review and extension of the group-theoretic approach has been given by Prince and Eliezer (1980). The problem of the time-dependent oscillator (1.1) and its invariants continues to attract lively attention and we single out the contributions made by Leach (1977), Lewis and Leach (1982), Wollenberg (1983) and Ray *et al* (1982).

We have studied the variable-mass oscillator with Hamiltonian

$$H(t)=\tfrac{1}{2}p^2/M(t)+\tfrac{1}{2}M(t)\omega_0^2q^2, \tag{1.4}$$

and we shall show that both systems (1.1) and (1.4) possess *two* equivalent families of linear and quadratic invariants. The alternative quadratic invariant for the variable-frequency system (1.1) will be shown to have the form

$$\tilde{I}(t)=p^2/\rho^2+\{M_0\rho q+[\dot{\rho}/\omega^2(t)]p\}^2, \tag{1.5}$$

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where ρ is any solution of the modified Pinney equation

$$\ddot{\rho} - 2(\dot{\omega}/\omega)\dot{\rho} + \omega^2(t)\rho = \omega^4(t)/\rho^3. \tag{1.6}$$

The relatively simple linear invariants do not seem to have attracted much attention, although the corresponding symmetry group generators for the system (1.1) have been displayed by several authors (e.g. Prince and Eliezer 1980, Ray *et al* 1982).

For simplicity of treatment we shall refer to classical mechanics. The extension to quantum mechanics is achieved formally by the simple procedure of symmetrising the product qp wherever it occurs. A fuller discussion is given by Wollenberg (1980).

2. The variable-mass oscillator

The variable-mass oscillator is conveniently treated by rescaling the coordinate and momentum so that (with M_0 some constant mass)

$$Q(t) = [M(t)/M_0]^{1/2}q, \qquad P(t) = [M_0/M(t)]^{1/2}p. \tag{2.1}$$

The Hamiltonian (1.4) transforms to

$$K(t) = \tfrac{1}{2}P^2/M_0 + \tfrac{1}{2}M_0\omega_0^2Q^2 + \varepsilon(t)QP, \tag{2.2}$$

where the time occurs solely in $\varepsilon(t)$ which is given by

$$\varepsilon(t) = (1/2M)\,\mathrm{d}M/\mathrm{d}t \tag{2.3}$$

(Colegrave and Abdalla 1981, 1982). We note that in the particular case of an exponentially decaying (or growing) mass, K is constant since for any Hamiltonian H, $\partial H/\partial t = 0 \Rightarrow \mathrm{d}H/\mathrm{d}t = 0$.

The equations of motion are

$$\dot{Q} = \partial K/\partial P = P/M_0 + \varepsilon Q, \qquad \dot{P} = -\partial K/\partial Q = -M_0\omega_0^2Q - \varepsilon P. \tag{2.4a, b}$$

The decoupled equations for Q and P are

$$\ddot{Q} + \Omega_Q^2(t)Q = 0, \qquad \Omega_Q^2(t) = \omega_0^2 - \varepsilon^2 - \dot{\varepsilon}, \tag{2.5a}$$

$$\ddot{P} + \Omega_P^2(t)P = 0, \qquad \Omega_P^2(t) = \omega_0^2 - \varepsilon^2 + \dot{\varepsilon}. \tag{2.5b}$$

Some examples of the variable-mass system are the following:

(a) A damped oscillator: $M(t) = M_0\exp(-2\Gamma t) \Rightarrow \varepsilon = -\Gamma$, $\Omega_Q^2 = \Omega_P^2 = \omega_0^2 - \Gamma^2$ (constant).

(b) A strongly pulsating oscillator: $M(t) = M_0\cos^2\nu t \Rightarrow \varepsilon = -\nu\tan\nu t$, $\Omega_Q^2 = \omega_0^2 + \nu^2$ (constant), $\Omega_P^2 = \Omega_Q^2 - 2\nu^2\sec^2\nu t$.

(c) Another strongly pulsating oscillator: $M(t) = M_0\sec^2\nu t \Rightarrow \varepsilon = \nu\tan\nu t$, $\Omega_P^2 = \omega_0^2 + \nu^2$ (constant), $\Omega_Q^2 = \Omega_P^2 - 2\nu^2\sec^2\nu t$.

(d) A single-pulse oscillator: $M(t) = M_0\,\mathrm{sech}^2\nu t \Rightarrow \varepsilon = -\nu\tanh\nu t$, $\Omega_P^2 = \omega_0^2 - \nu^2$ (constant), $\Omega_Q^2 = \Omega_P^2 + 2\nu^2\,\mathrm{sech}^2\nu t$.

(e) A moderately pulsating oscillator: $M(t) = M_0\exp(2\mu\sin\nu t) \Rightarrow \varepsilon = \mu\nu\cos\nu t$, $\Omega_Q^2 = \omega_0^2 - \mu^2\nu^2\cos^2\nu t + \mu\nu^2\sin\nu t$, $\Omega_P^2 = \omega_0^2 - \mu^2\nu^2\cos^2\nu t - \mu\nu^2\sin\nu t$.

We note that in (b) or (c) Ω_P^2 (or Ω_Q^2) changes sign periodically, but this does not unduly complicate the calculation of P (or Q), since Q (or P) is calculated first and then equation (2.4a) (or (2.4b)) is used to calculate the second dynamical variable (Colegrave and Abdalla 1982).

2.1. Linear invariants

We begin by seeking a first-degree invariant

$$J(t)=\lambda(t)Q+\mu(t)P. \tag{2.6}$$

We require (Goldstein 1980)

$$\dot{J}=\partial J/\partial t+(\partial J/\partial Q)\,\partial K/\partial P-(\partial J/\partial P)\,\partial K/\partial Q=0, \tag{2.7}$$

from which we see that λ and μ must satisfy

$$\dot{\lambda}=M_0\omega_0^2\mu-\lambda\varepsilon, \qquad \dot{\mu}=-\lambda/M_0+\mu\varepsilon. \tag{2.8a, b}$$

Eliminating μ we find

$$\ddot{\lambda}+\Omega_P^2(t)\lambda=0, \qquad \Omega_P^2(t)=\omega_0^2-\varepsilon^2+\dot{\varepsilon}. \tag{2.9}$$

Let us denote any solution of (2.9) by $\lambda=\rho^0(t)$; then we use (2.8a) to find μ and the invariant (2.6) assumes the form

$$J^{(P)}=\rho^0Q+(\dot{\rho}^0+\varepsilon\rho^0)P/(M_0\omega_0^2). \tag{2.10}$$

Again, eliminating λ from (2.8), we find

$$\ddot{\mu}+\Omega_Q^2(t)\mu=0, \qquad \Omega_Q^2(t)=\omega_0^2-\varepsilon^2-\dot{\varepsilon}. \tag{2.11}$$

We denote any solution of (2.11) by $\mu=\sigma^0(t)$ and solve (2.8b) for λ; then (2.6) gives a second invariant

$$J^{(Q)}=\sigma^0P-M_0(\dot{\sigma}^0-\varepsilon\sigma^0)Q. \tag{2.12}$$

We note the similarity between equations (2.4) and (2.8) and also between (2,5a, b) and (2.9), (2.11). The exchange transformation

$$Q\leftrightarrow\mu, \qquad P\leftrightarrow-\lambda, \tag{2.13}$$

changes the equations for Q, P into those for λ, μ and changes the sign of J.

2.2. Quadratic invariants

In a similar way we seek a second-degree invariant

$$I(t)=\alpha(t)Q^2+\beta(t)P^2+2\gamma(t)QP, \tag{2.14}$$

which requires

$$\dot{\alpha}=2M_0\omega_0^2\gamma-2\varepsilon(t)\alpha, \qquad \dot{\beta}=-2\gamma/M_0+2\varepsilon(t)\beta, \tag{2.15a, b}$$

$$\dot{\gamma}=-\alpha/M_0+M_0\omega_0^2\beta. \tag{2.15c}$$

A first integral of equations (2.15) is obtained by multiplying (2.15a) by β and (2.15b) by α and adding. This leads to

$$\alpha\beta=\gamma^2+A, \tag{2.16}$$

where A is a constant (cf Wollenberg 1980, theorem 2). Taking $A=0$ leads to the linear invariants of § 2.1.

Let us substitute in (2.15b) for β from (2.16) and for γ from (2.15a). This leads to

$$\tfrac{1}{2}(\ddot{\alpha}-\dot{\alpha}^2/2\alpha)=-\Omega_P^2(t)\alpha+M_0^2\omega_0^4A/\alpha. \tag{2.17}$$

Let us write $a = M_0^2\omega_0^4 A$ and $\rho(t) = a^{-1/4}\alpha^{1/2}(t)$; then (2.17) reduces to

$$\ddot{\rho} + \Omega_P^2(t)\rho = 1/\rho^3, \qquad \Omega_P^2(t) = \omega_0^2 - \varepsilon^2 + \dot{\varepsilon}. \tag{2.18}$$

The problem is now identifiable with that of Lewis and Riesenfeld (1969) in connection with the variable-frequency oscillator described by the Hamiltonian (1.1). With ρ *any* solution of (2.18) an invariant for the variable mass system (1.4) is

$$M_0^2\omega_0^4 a^{-1/2} I^{(P)} = [M_0\omega_0^2\rho Q + (\dot{\rho} + \varepsilon\rho)P]^2 + P^2/\rho^2. \tag{2.19}$$

A second family of invariants may be found by following a similar procedure, first eliminating α and solving for β. This leads to

$$\tfrac{1}{2}(\ddot{\beta} - \dot{\beta}^2/2\beta) = -\Omega_Q^2(t)\beta + A/(M_0^2\beta). \tag{2.20}$$

With $b = A/M_0^2$ and $\sigma(t) = b^{-1/4}\beta^{1/2}(t)$ equation (2.20) transforms to

$$\ddot{\sigma} + \Omega_Q^2(t)\sigma = 1/\sigma^3, \qquad \Omega_Q^2(t) = \omega_0^2 - \varepsilon^2 - \dot{\varepsilon}. \tag{2.21}$$

Thus with σ *any* solution of (2.21) a second quadratic invariant is

$$M_0^{-2} b^{-1/2} I^{(Q)} = Q^2/\sigma^2 + [(\dot{\sigma} - \varepsilon\sigma)Q - \sigma P/M_0]^2. \tag{2.22}$$

We notice the similarity with the J invariants of § 2.1. The relation between the functions ρ and ρ^0 is (Lewis 1968, Prince and Eliezer 1980, Wollenberg 1983)

$$\rho = [l(\rho_1^0)^2 + m(\rho_2^0)^2 + 2n\rho_1^0\rho_2^0]^{1/2}, \tag{2.23}$$

where ρ_1^0, ρ_2^0 are linearly independent solutions of equations (2.9), l, m, n are constants such that

$$lm - n^2 = W(\rho_1^0, \rho_2^0)^{-2} \tag{2.24}$$

and W is the Wronskian. We have a similar relation between σ and σ^0.

We shall discuss the connection between $\{I^{(Q)}\}$ and $\{I^{(P)}\}$ in § 4.

3. The variable-frequency system

We are now in a position to discuss Lewis and Riesenfeld's problem defined by the Hamiltonian (1.1). The Hamilton equations are

$$\dot{q} = \partial H/\partial p = p/M_0, \qquad \dot{p} = -\partial H/\partial q = -M_0\omega^2(t)q. \tag{3.1a, b}$$

The separate equations for q and p are

$$\ddot{q} + \omega^2(t)q = 0, \qquad \ddot{p} - 2(\dot{\omega}/\omega)\dot{p} + \omega^2(t)p = 0. \tag{3.2a, b}$$

We can make (3.2b) fit into the pattern of (2.5b) by writing

$$P = p/\omega(t), \tag{3.2c}$$

so that

$$\ddot{P} + \omega_P^2(t)P = 0, \qquad \omega_P^2(t) = \omega^2(t) + \ddot{\omega}/\omega - 2(\dot{\omega}/\omega)^2. \tag{3.2d}$$

3.1. Linear invariants

We seek an invariant

$$J = \lambda(t)q + \nu(t)p; \tag{3.3}$$

then using the Hamiltonian (1.1) we find

$$\dot{\lambda} = M_0\omega^2(t)\nu, \qquad \dot{\nu} = -\lambda/M_0. \tag{3.4}$$

Eliminating λ gives

$$\ddot{\nu} + \omega^2(t)\nu = 0 \tag{3.5}$$

and eliminating ν gives

$$\ddot{\lambda} - 2(\dot{\omega}/\omega)\dot{\lambda} + \omega^2(t)\lambda = 0 \tag{3.6a}$$

or, with $\mu = \lambda/\omega(t)$ (cf equations (3.2*c*, *d*)),

$$\ddot{\mu} + \omega_P^2(t)\mu = 0, \qquad \omega_P^2 = \omega^2(t) + \ddot{\omega}/\omega - 2(\dot{\omega}/\omega)^2. \tag{3.6b}$$

We notice again that the transformation (cf (2.13))

$$q \leftrightarrow \nu, \qquad p \leftrightarrow -\lambda, \tag{3.7}$$

changes (3.1) into (3.4) and makes $J \to -J$. Let σ^0, ρ^0 denote any solutions of (3.5). (3.6*a*) respectively; then linear invariants (3.3) analogous to (2.10) and (2.12) are

$$J^{(p)} = \rho^0 q + \dot{\rho}^0 p/[M_0\omega^2(t)], \qquad J^{(q)} = \sigma^0 p - M_0\dot{\sigma}^0 q. \tag{3.8a, b}$$

Alternatively, let τ^0 be any solution of (3.6*b*); then we may rewrite (3.8*a*) in the form (with $P = p/\omega(t)$ as in (3.2*c*))

$$J^{(P)} = \omega(t)\tau^0 q + [\dot{\tau}^0 + (\dot{\omega}/\omega)\tau^0]P/M_0. \tag{3.8c}$$

3.2. *Quadratic invariants*

For a quadratic invariant

$$I = \alpha(t)q^2 + \beta(t)p^2 + 2\gamma(t)qp \tag{3.9}$$

associated with the Hamiltonian (1.1) we find (cf Lewis and Riesenfeld 1969)

$$\dot{\alpha} = 2M_0\omega^2(t)\gamma, \qquad \dot{\beta} = -2\gamma/M_0, \tag{3.10a, b}$$

$$\dot{\gamma} = -\alpha/M_0 + M_0\omega^2(t)\beta. \tag{3.10c}$$

Again, as noted by Wollenberg (1980), a first integral exists of the form

$$\alpha\beta = \gamma^2 + B. \tag{3.11}$$

Following Lewis and Risenfeld (1969) and putting

$$\beta = c^{1/2}\rho^2, \qquad c = M_0^2 B, \tag{3.12}$$

we obtain equation (1.3) and the invariant (1.2) which we may write in the form

$$M_0^{-2}c^{-1/2}I^{(q)} = q^2/\sigma^2 + (\dot{\sigma}q - \sigma p/M_0)^2. \tag{3.13}$$

However, if we eliminate β and solve for α by setting

$$\alpha = M_0 c^{1/2}\rho^2 \tag{3.14}$$

we obtain equation (1.6) and the invariant (1.5). Alternatively, if we write

$$\mu(t) = \rho(t)/\omega(t), \tag{3.15}$$

then we obtain Pinney's equation (cf equations (3.2*d*) and (3.6*b*))

$$\ddot{\mu}+\omega_p^2(t)\mu=1/\mu^3,\qquad \omega_p^2=\omega^2(t)+\ddot{\omega}/\omega-2(\dot{\omega}/\omega)^2. \qquad (3.16)$$

The associated invariant (1.5) can be written in the alternative forms

$$\begin{aligned} M_0c^{-1/2}I^{(p)} &= p^2/\rho^2+\{M_0\rho q+[\dot{\rho}/\omega^2(t)]p\}^2 \\ &= P^2/\tau^2+\{M_0\omega(t)\tau q+[\dot{\tau}+(\dot{\omega}/\omega)\tau]P\}^2, \end{aligned} \qquad (3.17)$$

where $P=p/\omega(t)$ and τ is *any* solution of (3.16). The connection between τ and τ^0 is as described for ρ and ρ^0 in (2.23).

4. Connections between the invariants

We return to the case of variable mass discussed in § 2. The equations for the scaled coordinate Q and momentum P are more symmetrical than the equations for q and p in § 3, and consequently the analysis is tidier. However, our discussion can be extended quite easily to the variable-frequency case.

4.1. Exchange symmetry

We multiply $J^{(P)}$ given by (2.10) by a factor so that it has the same dimensions as $J^{(Q)}$ given by (2.12); thus

$$\bar{J}^{(P)}=M_0\omega_0\rho^0(t)Q+(\dot{\rho}^0+\varepsilon\rho^0)P/\omega_0,\qquad \bar{J}^{(Q)}=\sigma^0(t)P-M_0(\dot{\sigma}^0-\varepsilon\sigma^0)Q, \quad (4.1a, b)$$

where

$$\ddot{\rho}^0+\Omega_P^2(t)\rho^0=0,\qquad \Omega_P^2=\omega_0^2-\varepsilon^2+\dot{\varepsilon}, \qquad (4.1c)$$

$$\ddot{\sigma}^0+\Omega_Q^2(t)\sigma^0=0,\qquad \Omega_Q^2=\omega_0^2-\varepsilon^2-\dot{\varepsilon}. \qquad (4.1d)$$

Similarly, we shall take the second-degree invariants (2.19), (2.22) to have the same dimensions:

$$\bar{I}^{(P)}=P^2/\rho^2+[M_0\omega_0^2\rho Q+(\dot{\rho}+\varepsilon\rho)P]^2, \qquad (4.2a)$$

$$\bar{I}^{(Q)}=(M_0\omega_0)^2\{Q^2/\sigma^2+[(\dot{\sigma}-\varepsilon\sigma)Q-\sigma P/M_0]^2\}, \qquad (4.2b)$$

where

$$\ddot{\rho}+\Omega_P^2(t)\rho=1/\rho^3,\qquad \ddot{\sigma}+\Omega_Q^2(t)\sigma=1/\sigma^3. \qquad (4.2c, d)$$

Under the exchange transformation

$$Q\to Q'=(M_0\omega_0)^{-1}P,\qquad P\to P'=-M_0\omega_0 Q, \qquad (4.3)$$

the Hamiltonian (2.2)

$$K=(2M_0)^{-1}P^2+\tfrac{1}{2}M_0\omega_0^2Q^2+\varepsilon(t)QP\to\tfrac{1}{2}M_0\omega_0^2Q'^2+(2M_0)^{-1}P'^2-\varepsilon(t)Q'P' \qquad (4.4)$$

so that

$$\varepsilon(t)\to\varepsilon'(t)=-\varepsilon(t), \qquad (4.5a)$$

$$\Omega_Q^2(t)\to\Omega_{P'}^2(t),\qquad \Omega_P^2(t)\to\Omega_{Q'}^2(t), \qquad (4.5b)$$

$$\sigma^0(t)\to\rho^0(t),\qquad \rho^0(t)\to\sigma^0(t), \qquad (4.5c)$$

$$\bar{J}^{(P)} \to -\bar{J}^{(Q')}, \qquad \bar{J}^{(Q)} \to \bar{J}^{(P')}, \tag{4.5d}$$

$$\bar{I}^{(P)} \to \bar{I}^{(Q')}, \qquad \bar{I}^{(Q)} \to \bar{I}^{(P')}. \tag{4.5e}$$

4.2. *Linear invariants*

We may write the general solution of (2.8*a*, *b*) in the form

$$\lambda \equiv \rho^0 = M_0\omega_0 a_0(\rho_1^0 + k\rho_2^0), \tag{4.6a}$$

$$\mu \equiv \sigma^0 = a_0[(\dot{\rho}_1^0 + \varepsilon\rho_1^0) + k(\dot{\rho}_2^0 + \varepsilon\rho_2^0)]/\omega_0, \tag{4.6b}$$

where a_0, k are arbitrary constants and ρ_1^0, ρ_2^0 are independent solutions of $\ddot{\rho} + \Omega_P^2(t)\rho = 0$. Alternatively we may write

$$\lambda \equiv \rho^0 = -M_0 b_0[(\dot{\sigma}_1^0 - \varepsilon\sigma_1^0) + k'(\dot{\sigma}_2^0 - \varepsilon\sigma_2^0)], \tag{4.7a}$$

$$\mu \equiv \sigma^0 = b_0(\sigma_1^0 + k'\sigma_2^0), \tag{4.7b}$$

where b_0, k' are arbitrary constants and σ_1^0, σ_2^0 are independent solutions of $\ddot{\sigma} + \Omega_Q^2(t)\sigma = 0$. Obviously the invariants (4.1*a*, *b*) interchange on switching from one of the alternatives (4.6), (4.7) to the other. The constants k, k' provide parametrisations for $J^{(P)}, J^{(Q)}$ respectively and it is clear that a one–one correspondence exists between k, k' such that (apart from a possible multiplicative factor) $J^{(P)}(k) = J^{(Q)}(k')$, i.e. the two families $\{J^{(P)}(k)\}$, $\{J^{(Q)}(k')\}$ coincide.

A linear invariant is a certain linear combination of the initial (scaled) position and momentum. As an example, let us take the case of damping ($\varepsilon = -\Gamma$, constant); then with $k = 0$, $a_0 = 1$, $\rho_1^0 = \cos\omega t$ ($\omega^2 = \omega_0^2 - \Gamma^2$) in (4.6)

$$\begin{aligned}\bar{J}^{(P)}(k=0) &\equiv M_0\omega_0 \cos\omega t Q(t) - (\omega \sin\omega t + \Gamma\cos\omega t)P(t)/\omega_0 \\ &= M_0\omega_0 Q(0) - (\Gamma/\omega_0)P(0).\end{aligned} \tag{4.8}$$

4.3. *Quadratic invariants*

Let us consider the general solution of the linear equations (2.15). We note that:

(i) if $\alpha(t)$, $\beta(t)$, $\gamma(t)$ is a solution, then so is $z\alpha(t)$, $z\beta(t)$, $z\gamma(t)$, where z is any constant,

(ii) apart from an arbitrary multiplicative constant as in (i) the solution is unique (Coddington 1961).

In our first method of solution (eliminating β), with $\alpha_1(t) = M_0\omega_0^2 A_1^{1/2}\rho^2(t)$ any solution of (2.17) and A_1 ($\neq 0$) a choice of A in (2.16), we calculate from (2.15*a*, *c*)

$$\gamma_1(t) = (2M_0\omega_0^2)^{-1}(\dot{\alpha}_1 + 2\varepsilon\alpha_1), \tag{4.9a}$$

$$\beta_1(t) = (M_0\omega_0^2)^{-2}[\dot{\alpha}_1^2/(4\alpha_1) + \varepsilon\dot{\alpha}_1 + \varepsilon^2\alpha_1 + A_1 M_0^2\omega_0^4/\alpha_1]. \tag{4.9b}$$

On differentiating (4.9*b*) once and twice, using (2.17) to express $\ddot{\alpha}_1$ in terms of $\dot{\alpha}_1$ and α_1, we can establish that β_1 satisfies (2.20). Our second method (eliminating α) gives, with $\beta_2(t) = A_2^{1/2}\sigma^2(t)/M_0$ any solution of (2.20) and A_2 ($\neq 0$) a choice of A in (2.16),

$$\gamma_2(t) = -\tfrac{1}{2}M_0(\dot{\beta}_2 - 2\varepsilon\beta_2), \tag{4.10a}$$

$$\alpha_2(t) = M_0^2[\dot{\beta}_2^2/(4\beta_2) - \varepsilon\dot{\beta}_2 + \varepsilon^2\beta_2 + A_2/(M_0^2\beta_2)]. \tag{4.10b}$$

Again we may check that α_2 satisfies (2.17) in accordance with the uniqueness theorem (ii) above or, equivalently, the characterisation of Wollenberg (1980).

Let us write the invariants $\bar{I}^{(P)}$, $\bar{I}^{(Q)}$ of (4.2a, b) in the form

$$\bar{I}^{(P)} = \bar{\alpha}_1(t)Q^2 + \bar{\beta}_1(t)P^2 + 2\bar{\gamma}_1(t)QP, \tag{4.11a}$$

$$\bar{I}^{(Q)} = \bar{\alpha}_2(t)Q^2 + \bar{\beta}_2(t)P^2 + 2\bar{\gamma}_2(t)QP; \tag{4.11b}$$

then from (2.19), (2.22) we see that

$$\bar{I}^{(P)} = M_0\omega_0^2 A_1^{-1/2} I^{(P)}, \qquad \bar{I}^{(Q)} = M_0\omega_0^2 A_2^{-1/2} I^{(Q)}. \tag{4.12}$$

Also, from (4.2)

$$\bar{\alpha}_1(t) = M_0^2\omega_0^4\rho^2(t), \qquad \bar{\beta}_2(t) = \omega_0^2\sigma^2(t). \tag{4.13}$$

Let us suppose that $\bar{I}^{(P)} = z\bar{I}^{(Q)}$ where z is a dimensionless constant; then equating the coefficients of Q^2, P^2 and QP in (4.2a, b) gives

$$\omega_0^2\rho^2 = z[1/\sigma^2 + (\dot{\sigma} - \varepsilon\sigma)^2], \qquad \omega_0^2\sigma^2 = (1/z)[1/\rho^2 + (\dot{\rho} + \varepsilon\rho)^2] \tag{4.14a, b}$$

$$\rho(\dot{\rho} + \varepsilon\rho) = -z\sigma(\dot{\sigma} - \varepsilon\sigma), \tag{4.14c}$$

where we must remember that ρ, σ satisfy the Pinney equations (2.18), (2.21). In § 5 we shall solve equations (4.14) in the case of a damped oscillator.

5. The case of damping

We consider example (a) of § 2. This is the only case in which $\Omega_P = \Omega_Q$. Thus $\varepsilon = -\Gamma$ and

$$\Omega_P = \Omega_Q = \omega = (\omega_0^2 - \Gamma^2)^{1/2}. \tag{5.1}$$

In this case the Pinney equations (2.18), (2.21) admit the first integrals

$$\dot{\rho}^2 + \omega^2\rho^2 + 1/\rho^2 = C\omega, \qquad \dot{\sigma}^2 + \omega^2\sigma^2 + 1/\sigma^2 = D\omega, \tag{5.2a, b}$$

where C, D are dimensionless positive constants. Equations (4.14), (5.2) reduce to

$$C = zD, \qquad \omega(\rho^2 + z\sigma^2) + (\Gamma/\omega)(\rho\dot{\rho} - z\sigma\dot{\sigma}) = C, \tag{5.3a, b}$$

$$\rho\dot{\rho} + z\sigma\dot{\sigma} = \Gamma(\rho^2 - z\sigma^2). \tag{5.3c}$$

Let us write $X = \rho^2 + z\sigma^2$, $Y = \rho^2 - z\sigma^2$; then (5.3b, c) become

$$\omega X + \tfrac{1}{2}(\Gamma/\omega)\dot{Y} = C, \qquad \tfrac{1}{2}\dot{X} = \Gamma Y. \tag{5.4a, b}$$

Hence Y satisfies the equation

$$\ddot{Y} + 4\omega^2 Y = 0. \tag{5.4c}$$

From (5.4) we may easily calculate ρ^2 and $z\sigma^2$; hence from (4.9a), (4.10a) and (4.13) we calculate the coefficients in $\bar{I}^{(P)}$, $\bar{I}^{(Q)}$ given by (4.11a, b). These are

$$\begin{aligned}\bar{\alpha}_1(t) &= z\bar{\alpha}_2(t)\\ &= \tfrac{1}{2}M_0^2\omega_0^4\{C/\omega + [2\rho^2(0) - C/\omega]\cos 2\omega t\\ &\quad + (\Gamma\omega)^{-1}[C\omega - (\omega_0^2 - 2\Gamma^2)\rho^2(0) - z\omega_0^2\sigma^2(0)]\sin 2\omega t\},\end{aligned} \tag{5.5a}$$

$$\bar{\beta}_1(t) = z\bar{\beta}_2(t)$$
$$= \tfrac{1}{2}\omega_0^2\{C/\omega + [2z\sigma^2(0) - C/\omega]\cos 2\omega t$$
$$- (\Gamma\omega)^{-1}[C\omega - (\omega_0^2 - 2\Gamma^2)z\sigma^2(0) - \omega_0^2\rho^2(0)]\sin 2\omega t\}, \quad (5.5b)$$

$$\bar{\gamma}_1(t) = z\bar{\gamma}_2(t)$$
$$= -\tfrac{1}{2}[M_0\omega_0^4/(\omega\Gamma)]\{C\Gamma^2/\omega_0^2 - \omega[C/\omega - \rho^2(0) - z\sigma^2(0)]\cos 2\omega t$$
$$+ \Gamma[\rho^2(0) - z\sigma^2(0)]\sin 2\omega t\}. \quad (5.5c)$$

Taking $\rho_1^0 = \sigma_1^0 = \cos\omega t$, $\rho_2^0 = \sigma_2^0 = \sin\omega t$, equations (2.23), (2.24) give

$$\rho^2(t) = l\cos^2\omega t + m\sin^2\omega t + 2n\cos\omega t\sin\omega t, \qquad lm - n^2 = \omega^{-2}, \quad (5.6a)$$

$$\sigma^2(t) = l'\cos^2\omega t + m'\sin^2\omega t + 2n'\cos\omega t\sin\omega t, \qquad l'm' - n'^2 = \omega^{-2}. \quad (5.6b)$$

Comparing equations (5.5*a*, *b*) and (5.6*a*, *b*) we see that

$$l = \rho^2(0), \qquad m = C/\omega - \rho^2(0), \qquad n = (2\Gamma\omega)^{-1}[C\omega - (\omega_0^2 - 2\Gamma^2)\rho^2(0) - z\omega_0^2\sigma^2(0)], \quad (5.7a)$$

$$l' = \sigma^2(0), \qquad zm' = C/\omega - z\sigma^2(0),$$

$$zn' = -(2\Gamma\omega)^{-1}[C\omega - (\omega_0^2 - 2\Gamma^2)z\sigma^2(0) - \omega_0^2\rho^2(0)]. \quad (5.7b)$$

It is now quite clear that to satisfy $lm - n^2 = l'm' - n'^2 = \omega^{-2}$, we must choose $z = 1$ and (with $\rho^2(0)$, $\sigma^2(0)$ arbitrary) C a root of the quadratic equation

$$\omega^2C^2 - 2C\omega\omega_0^2[\rho^2(0) + \sigma^2(0)] + \omega_0^4[\rho^2(0) + \sigma^2(0)]^2 - 4\omega_0^2\Gamma^2\rho^2(0)\sigma^2(0) + 4\Gamma^2 = 0. \quad (5.8)$$

An explicit form for any quadratic invariant $\bar{I}^{(P)} = \bar{I}^{(Q)}$ for the damped oscillator results when we insert a value for C from (5.8), together with $z = 1$, into (5.5), (4.11).

5.1. *The Hamiltonian as a special quadratic invariant*

As a special case we choose

$$\rho^2(0) = \sigma^2(0) = C/(2\omega); \quad (5.9a)$$

then equation (5.8) requires

$$C = 2. \quad (5.9b)$$

From (5.5) we see that $\bar{\alpha}$, $\bar{\beta}$, $\bar{\gamma}$ remain constant and the corresponding invariant is

$$\bar{I} = 2M_0\omega_0^2K/\omega, \quad (5.10a)$$

where

$$K = \tfrac{1}{2}M_0\omega_0^2Q^2 + \tfrac{1}{2}P^2/M_0 - \Gamma QP \quad (5.10b)$$

is the canonical Hamiltonian derived by Colegrave and Abdalla (1981). We note that equations (5.9) describe the special solution $\rho = \omega^{-1/2}$ of the Pinney equation $\ddot{\rho} + \omega^2\rho = \rho^{-3}$.

5.2. *Parametrisation for $\bar{I}^{(P)}$, $\bar{I}^{(Q)}$*

We may parametrise $\bar{I}^{(P)} = \bar{I}^P(m, n)$, $\bar{I}^Q = \bar{I}^{(Q)}(m', n')$ from (4.2), (5.6) with

$$l = (n^2 + \omega^{-2})/m, \qquad l' = (n'^2 + \omega^{-2})/m'. \quad (5.11)$$

Since $\{\bar{I}^{(P)}\} = \{\bar{I}^{(Q)}\}$ it is clear that a bijective correspondence $m, n \to m', n'$ must exist between the parameters.

5.3. The case of constant mass

When $\Gamma \to 0$ equations (5.3) give (with $z = 1$)

$$\rho^2 + \sigma^2 = C/\omega_0. \tag{5.12}$$

Looking at equations (5.6) we see that this requires

$$l + l' = m + m', \qquad n + n' = 0. \tag{5.13}$$

Using (5.11), (5.13) we have the bijective correspondence

$$m' = (n^2 + \omega_0^{-2})/m, \qquad n' = -n. \tag{5.14}$$

5.4. Connections between the linear and quadratic invariants

Taking $A = 0$ in (2.16) leads to the simple connection $I = J^2$ between the quadratic invariant I and a certain invariant J. Since there cannot be more than two independent functions of two variables, it follows that when $A \neq 0$

$$I = aJ_1^2 + bJ_2^2 + 2cJ_1J_2, \tag{5.15}$$

where a, b, c are constants and J_1, J_2 are two independent linear invariants.

6. Conclusion

The time-independent oscillator has the group structure associated with equations (2.9) and (2.18) with $\varepsilon(t) = 0$ (Wulfman and Wybourne 1976). We have shown that the invariants, and hence the symmetries, of the variable-mass oscillator are described by the two sets of similar equations (2.9), (2.18) and (2.11), (2.21) which are associated with the equations of motion for the (scaled) momentum and coordinate respectively. The connection between the constants of the motion and the group SL(3, ℝ) of the one-dimensional time-dependent oscillator is discussed fully by Prince and Eliezer (1980). Group-theoretical methods are discussed also by Wolf (1981).

We have shown that the family of linear invariants $\{J^{(P)}\}$ coincides with the family $\{J^{(Q)}\}$ for all functions $\varepsilon(t)$; similarly the family of quadratic invariants $\{I^{(P)}\}$ coincides with the family $\{I^{(Q)}\}$ and obviously the quadratic invariants are quadratic functions of the linear ones. The alternative forms for the quadratic invariants have enabled us to calculate explicit results in the case of exponentially damped (or constant) mass. In this case the Hamiltonian is a special quadratic invariant.

We hope to extend the explicit formulae for $I^{(P)}$ or $I^{(Q)}$ to the strongly pulsating oscillator (Colegrave and Abdalla 1982) and possibly to other cases. We feel, too, that some interesting applications can be made of the lowering and raising operators, a, $a^\dagger$, introduced by Lewis and Riesenfeld (1969), in terms of which a quadratic invariant may be written

$$I = \hbar(a^\dagger a + \tfrac{1}{2}), \qquad [a, a^\dagger] = 1.$$

The case of variable frequency (Lewis and Riesenfeld 1969, Wollenberg 1980, Ray *et al* 1982) leads to corresponding linear and quadratic invariants, as discussed in § 3.

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